OLM 9.10* Area of a parallelogram

TBox 9.1 of the book relies on the following statement:

The area of a parallelogram, spanned by two vectors, equals to the absolute value of the determinant composed of the two vectors.

The justification is as follows. From geometry, the area reads, as

$$A = |\boldsymbol{a}||\boldsymbol{b}| \sin \varphi,$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are the two vectors and φ is the angle between them. The difficulty lies in determining φ algebraically, as a function of the vectors. We know that the scalar product of the vectors is also related to the angle:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 = |\boldsymbol{a}||\boldsymbol{b}| \cos \varphi.$$

Also, sine and cosine are related (Pythagorean theorem):

$$\sin^2 \varphi + \cos^2 \varphi = 1.$$

It follows for the square of the area

$$A^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2 = (a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1 b_1 + a_2 b_2)^2 =$$

$$= a_1^2 b_2^2 + a_2^2 b_1^2 - 2 a_1 a_2 b_1 b_2.$$

On the other hand, the square of the determinant reads, as

$$\left(\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\right)^2 = (a_1 b_2 - a_2 b_1)^2 = a_1^2 b_2^2 + a_2^2 b_1^2 - 2 a_1 a_2 b_1 b_2,$$

which is just the same as the square of the area above. This proves the statement. □

Note that the sign of the determinant depends on the relative directions of the vectors. Exchanging the vectors changes the sign of the determinant, while the area remains positive. (The determinant is positive, if the first vector can be rotated into the direction of the second one with a positive, i.e., anti-clockwise, rotation.) This is why we should use the *absolute value* of the determinant for the area. In Figure 9.1 the relative directions, therefore the sign of the determinant, are different. This is important: positive determinant means stability, while the negative one implies an unstable fixed point. The area of the parallelogram does not reflect this difference.

Note also that the notation in TBox 9.1 is different from the one used here. In the TBox the matrix reads, as $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, which notation would not be convenient for the proof above.